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A Handy Analytical Approximate Solution for a Heat Transfer Problem Using the Method of Subdomains with Boundary Conditions

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ABSTRACT: The aim of this paper is to show the application of the method of weighted residuals through subdomains to offer a solution to the steady-state one-dimensional heat conduction problem in a slab with thermal conductivity linearly dependent on temperature. The proposed solution is a fourth-degree polynomial, derived using three subdomains. Despite its simplicity, the solution demonstrates good accuracy, as evidenced by an RMS error of 0.0009689460673930. However, if greater accuracy is required, the Method of Weighted Residuals allows for the use of more subdomains with a higher-degree polynomial.

KEYWORDS- Nonlinear differential equations, boundary value problems, heat problems.

I. INTRODUCTION

Thermodynamics, nestled within the domain of physics, is dedicated to scrutinizing heat-related phenomena. Its focal point revolves around elucidating the transformation of various energy forms and the intricate nexus between these transformations and temperature. Historically, the development of thermodynamics paralleled endeavors aimed at enhancing machine efficiency, striving to minimize energy dissipation in the form of heat [1, 2].

The foundational pillars of thermodynamics comprise a suite of laws delineating energy behavior. Foremost among them is the principle of conservation of energy, which mandates that energy can neither be created nor destroyed but can only undergo transmutation into another form. Thus, heat emerges as merely one facet of energy, stemming from other forms such as work. The second law of thermodynamics delineates the presence of escalating entropy within a closed system, wherein entropy represents a trajectory toward disorder rendering energy less usable. Lastly, the third law of thermodynamics posits the insurmountable barrier to attaining absolute zero temperature within a system through a finite number of steps [1, 2].

In addition to the aforementioned laws, the zero principle postulates that disparate thermal systems will attain equilibrium when in some form of contact. Thus, if system A interacts with system B, and system B interfaces with system C, eventual equilibrium ensues, characterized by identical temperatures among them. This principle intimates that the universe had a genesis and remains dynamic, as its failure to reach equilibrium implies ongoing processes of energy exchange and transformation [1, 2].

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Heat transfer exhibits both directionality and magnitude. The rate of heat conduction in a specific direction is directly proportional to the temperature gradient, which represents the rate of temperature change per unit distance along that direction. Generally, heat conduction within a medium occurs in three dimensions and is time-dependent. Thus, the temperature within a medium varies not only with time but also with spatial coordinates, denoted as $T = T(x, y, z, t)$. Heat conduction within a medium is termed steady when the temperature remains constant over time, and unsteady or transient when it fluctuates. It is considered one-dimensional when conduction primarily occurs along a single dimension, with negligible conduction along the other two primary dimensions. Similarly, it is termed two-dimensional when conduction along the third dimension is insignificant, and three-dimensional when significant conduction exists in all dimensions [1, 2].

The primary objective of this article is to develop a convenient analytical approximate solution for the boundary value problem associated with the steady-state one-dimensional conduction of heat within a slab. Notably, the thermal conductivity of this slab is assumed to exhibit a linear dependence on temperature. Given the fundamental significance of heat transfer phenomena, both in theoretical contexts and practical applications involving the design and operation of equipment, there exists a critical need to explore analytical approximate solutions for the equations governing these phenomena. This pursuit aligns with the broader research context and underscores the importance of developing efficient methods for addressing heat transfer problems.

There are several alternative semi-analytical methods to solve nonlinear differential equations, for example, power extender series method (PSEM) [3], Homotopy perturbation method (HPM) [4], homotopy analysis method (HAM) [5], Adomian's decomposition method [6], modified Taylor series method (MTM) [7-10], perturbation method (PM)[11], the methods of weighted residuals (MWR) [12,13,14], among many others. In this study, we assume that the exact solution for case study on one-dimensional (1-D) steady-state heat conduction is solved using Maple2021. Additionally, one of the metrics we will use to evaluate the precision of the proposed approximations is the root-mean-squared (RMS) error.

This paper is organized as follows. Section II introduces the bases of the weighted residuals method. Section III presents a case study, Section IV presents the discussion, and section V presents the conclusions.

II. WEIGHTED RESIDUALS METHOD

To set the stage for MWR, consider a general boundary value problem whose governing differential equation is given by

$$\begin{aligned} Lu(\mathbf{x}) &= 0, \qquad \mathbf{x} \in \Omega, \\ u(\mathbf{x}) &= g(\mathbf{x}), \qquad \mathbf{x} \in \Gamma, \end{aligned} \tag{1}$$

where L represents a differential operator, $u = u(x)$ denotes the dependent variable defined within a region Ω with boundary Γ, and x denotes spatial coordinates. In the Method of Weighted Residuals (MWR), the aim is to approximate the solution $u(x)$ of Equation (1) using a trial solution $u_n(x)$, which is selected in a particular manner. However, this trial solution typically doesn't satisfy the governing differential equation. Therefore, substituting the trial solution into the governing differential equation yields a residual, denoted by R [12-14]. To obtain the "best" solution, efforts are made to distribute this residual throughout the region Ω by minimizing the integral of the residual across Ω. This can be expressed as:

$$\text{Minimize} = \int_{\Omega} R d\Omega. \tag{2}$$

The scope of opportunities to achieve this objective can be expanded by ensuring that a weighted value of the residual is minimized across the entire region of interest. By using a weighting function, it is possible to achieve a minimum value of zero for the weighted integral. Letting the weighting functions be represented by w, the required objective of the MWR is then defined as follows:

$$\int_{\Omega} wR d\Omega = 0. \tag{3}$$

The concept of approximating the solution $u(x)$ of a differential equation using trial solutions is well-established. However, the proper selection of the trial solution is essential for the success MWR. This selection is powerful because it allows the incorporation of known information about the problem into the trial solution. In lower-order approximations (i.e., for small n in $u_n(x)$), this choice can significantly affect the accuracy of the results. In higher-order approximations, it can influence the convergence of the method [12-14].

Of all the trial solutions employed by several people, perhaps the polynomial series such as

$$u_n(\mathbf{x}) = \sum_{i=1}^{n} c_i N(\mathbf{x}) = \sum_{i=1}^{n} c_i x^i. \tag{4}$$

Polynomials are the most popular choice for this purpose. In equation (4), c_i are arbitrary constants that need to be determined during the minimization process described in equation (3). The functions $N(x)$ are preselected and are known as trial functions or shape functions. The widespread preference for polynomials is primarily due to their ease of manipulation. Furthermore, the weighting functions can be selected in various ways, with each choice corresponding to a different MWR criterion [12-14].

Subdomain method. In this process, the Ω domain is divided into m small subdomains of $\Omega_j, j = 1,2,\cdots,m$, which are not necessary disjoint. The weights are chosen as

$$w_j = \begin{cases} 1, & \mathbf{x} \in \Omega_j, \\ 0, & \mathbf{x} \notin \Omega_j, \end{cases} \tag{5}$$

and

$$\int_{\Omega_j} R d\Omega_j = 0, \qquad j = 1,2,\cdots m. \tag{6}$$

Indeed, as m increases, the differential equation integrated over each subdomain tends toward zero. Consequently, the equation is satisfied, on average, in smaller and smaller domains, ultimately approaching zero across the entire domain [12-14].

Colocation Method. In this method, the weighting functions w_j are chosen to be the displaced Dirac Delta functions

$$w_j = \delta_j = \delta(\mathbf{x} - \mathbf{x}_j). \tag{7}$$

Now (2) is given by

$$\int_{\Omega_j} w_j R d\Omega = \int \delta_j R d\Omega = R_j = 0, \qquad j = 1,2,\cdots m, \tag{8}$$

where R_j represents the value of R evaluated at the point $\mathbf{x}_j$. Consequently, the residual is enforced to vanish at m specified collocation points, $x_j = 1,2,\cdots m$. As m increases, the residual vanishes at more and more points, presumably approaching zero-every-where.

Least Squares Method. In this method, the weighting functions w_j are choosen to be

$$w_j = \frac{\partial R}{\partial c_j}. \tag{9}$$

Now Eq. (2) is given by

$$\frac{\partial}{\partial c_j} \int_{\Omega} R^2 d\Omega = 2 \int_{\Omega} \frac{\partial R}{\partial c_j} R d\Omega = 0, j = 1,2,\cdots n. \tag{10}$$

The integral of the square of the residual is minimized with respect to the undetermined parameters to provide N simultaneous equations for the c_j's.

Method of moments. In this method, the weighting functions w_j are choosen to be

$$w_j = P_j(\mathbf{x}).$$

(11)

Where $P_j(x)$ are orthogonal polynomials defined over the domain Ω.This procedure is particularly advantageous in one-dimensional problems, where the theory of orthogonal polynomials is well-established. In such problems, the widespread use of weighting functions, denoted by $w(x)$, leads to the following:

$$\int_{\Omega} x^j R d\Omega = 0.$$

(12)

The structure of Equation (12) gave rise to the term "method of moments". It's worth noting, however, that the set $\{w_j\} = \{x^j\} = \{1, x, x^2, \cdots\}$ is not orthogonal over the interval $0 \le x \le 1$, and typically, better results can be achieved by orthogonalizing them before application [12-14].

Galerkin method. In this method, the weighting functions w_j are choosen to be identical to the shape functions N_j themselves, that is,

$$w_j = N_j(\mathbf{x}), \qquad j = 1,2, \cdots, m.$$

(14)

Therefore, Eq. (3) is given by

$$\int_{\Omega} N_j(\mathbf{x}) R d\Omega = 0, \quad j = 1,2, \cdots, m.$$

(15)

In vector-matrix notation, we have

$$\int_{\Omega} N R d\Omega = 0\,,$$

(15)

where $\boldsymbol{N} = (N_1, N_2, \cdots, N_m)^T$. Utilizing the well-established fact that a continuous function is zero if it is orthogonal to every member of a set, it becomes apparent that the Galerkin method enforces the residual to be zero by ensuring its orthogonality to each member of a complete set of basis functions [12-14].

III. STUDY CASE

This article presents a case study on one-dimensional (1-D) steady-state heat conduction in a slab with linearly temperature-dependent thermal conductivity [9]; see Fig. 1.

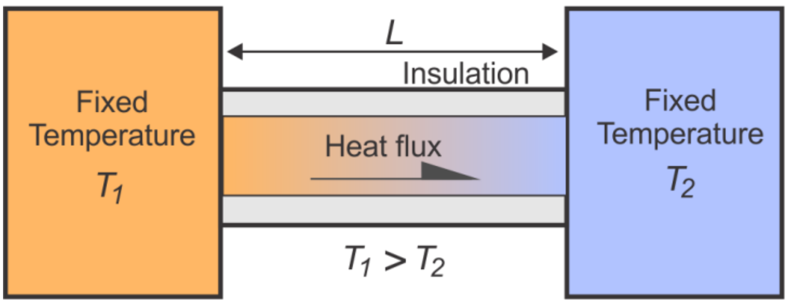

Figure 1. The 1-D conduction of heat through an insulated slab.

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The non-dimensionalization process for this problem was presented in [9]. Additionally, [9] provided an approximate solution using the MTSM. In this work, we will determine the approximate solution using the MWR, specifically the method of moments. The differential equation for this case study is given by

$$\frac{d^2y}{dz^2} + \varepsilon y \frac{d^2}{dz^2} + \varepsilon \left(\frac{dy}{dz}\right)^2 = 0, \tag{16}$$

with boundary conditions given by

$$y(0) = 1, \quad y(1) = 0. \tag{17}$$

In the case of $\varepsilon = 1$, the exact solution obtained with Maple 2021 becomes

$$y_E = -1 + \sqrt{-3x + 4}. \tag{18}$$

Applying the subdomain method, we have the subdomains $\Omega_1 = [0,0.33], \Omega_2 = [0.33,0.67], \Omega_3 = [0.67,1]$. Furthermore, the proposed solution is a polynomial of degree 4 given by

$$y = ex^4 + dx^3 + cx^2 + bx + a. \tag{19}$$

Substituting the boundary conditions in (19) we have

$$\mathrm{y} = ex^4 + dx^3 + cx^2 - (1 + c + d + e)x + 1. \tag{20}$$

Equation (20) is substituted in (16) to obtain the residue R. Applying (6) to integrate Rin the ohms subdomains, we obtain three equations, where $c, d, e,$ must be determined with a numerical algorithm such as Newton-Raphson [15,16].

$$0.00170473771908e^2 + 0.009040275783de + 0.0234812358ce + 0.0117406179d^2 + 0.05929605cd + 0.05929605e(-1 - c - d - e) + 0.071874c^2 + 0.143748d(-1 - c - d - e) + 0.287496e + 0.3267c(-1 - c - d - e) + 0.6534d + 1.320c + 0.330(-1 - c - d - e)^2 = 0,$$

$$0.24072372649384e^2 + 0.6241683994de + 0.7865938284ce + 0.3932969142d^2 + 0.94826cd + 0.94826e(-1. - c - d - e) + 0.529652c^2 + 1.059304d(-1. - c - d - e) + 2.118608e + 1.020c(-1. - c - d - e) + 2.040d + 1.360c + 0.340(-1. - c - d - e)^2 = 0,$$

$$3.75757153578708e^2 + 6.366791324817de + 5.1899249358ce + 2.5949624679d^2 + 3.99244395cd + 3.992443950e(-1. - c - d - e) + 1.398474c^2 + 2.796948d(-1. - c - d - e) + 5.593896e + 1.6533c(-1. - c - d - e) + 3.3066d + 1.320c + 0.330(-1. - c - d - ef)^2 = 0. \tag{21}$$

Solving equation system (21), the solutions are $c = -0.201853053750679, d = 0.141480597362956, e = -0.192072645639957$. Substituting the numerical values found in Eq.(19) the solution becomes

$$y_s = 1 - 0.747554897972x - 0.20185305375x^2 + 0.141480597363x^3 - 019207264564x^4. \tag{22}$$

The solution obtained with MTSM [19] is given by

$$y_T(z) = 1 - 0.768425x - 0.147619245x^2 - 0.056717159x^3 - 0.027239301x^4. \tag{23}$$

Figure 2 presents a comparison between the exact solution (18), MWR (22) and MTSM (23). Note that solution (22) presents better performance.

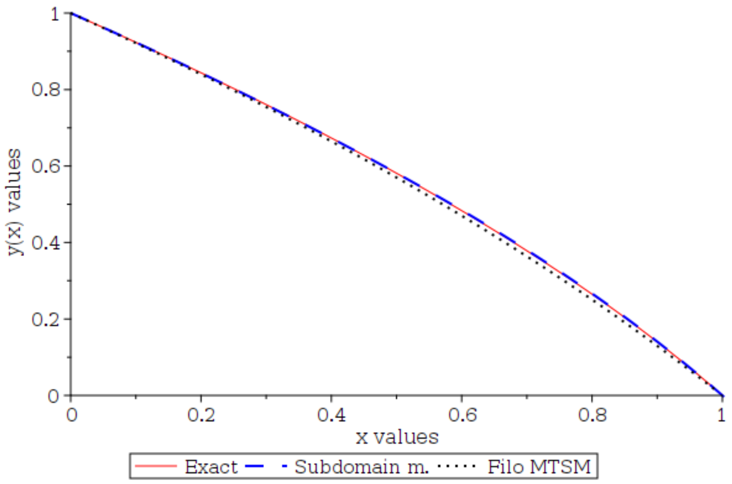

Figure 2. Comparison of exact solutions, subdomain methods and MTSM.

IV. DISCUSSION

To solve equation (16), three subdomains were employed. The resulting solution, equation (22), is a fourth-degree polynomial. Figure 2 compares the absolute errors of equations (22) and (23). The absolute error of solution (23), obtained using the MWR, is notably smaller. Notably, equation (23) is also a fourth-degree polynomial, determined using the MTSM [9].

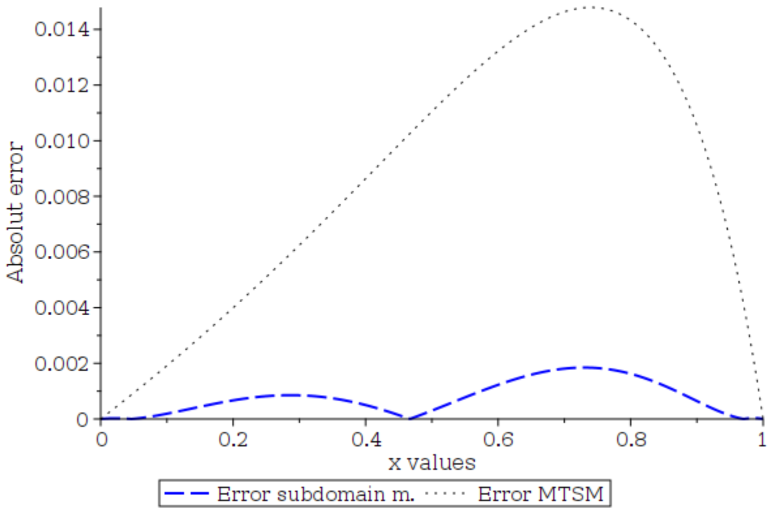

Figure 2. Absolute error for equations (21) and (22).

To measure the RMS error in the interval defined by the boundary conditions, we will use the formula

$$E_{rms} = \sqrt{\frac{1}{b-a}\int_a^b (E(t))^2 dt.} \tag{24}$$

Table 1 presents the RMS error for different values of ε, along with the approximate solutions obtained for each ε. As ε increases, the RMS error in the approximations also increases. Specifically, when $\varepsilon = 1$, the RMS error is 0.0009689460673930. In contrast, the RMS error obtained in equation (22) is 0.00980864863178778. This indicates that the RMS error using MWR is 10.123 times lower than that obtained using the MTSM [9]. Consequently, the solution obtained with MWR has greater accuracy than the solution obtained with MTSM.

Table 1: RMS error for different ε.

Value ε	Error RMS	Polynomial equation obtained
0.5	0.0001147954053900	$y_s = 1 - 0.8329973795x - 124982871724x^2 - 0.00154604869x^3 - 0.0404737x^4$
1	0.0009689460673930	$y_s = 1 - 0.7475548797x - 0.20185305375x^2 - 0.14148059736x^3 - 0.19207264564x^4$
1.5	0.0028905555956655	$y_s = 1 - 0.69340869794x - 0.3003555942x^2 - 0.41445902701x^3 - 0.42069473490622x^4$

If greater accuracy is desired, then more subdomains must be considered in the solution process [12-14].

V. CONCLUSIONS

The article employs the method of subdomains to derive a polynomial solution for the steady-state one-dimensional heat conduction problem in a slab with a thermal conductivity linearly dependent on temperature, under Dirichlet boundary conditions. This method entails utilizing a trial function integrated across subdomains following a prescribed methodology. The resultant system of equations is expressed in terms of constants that require determination through numerical algorithms like Newton-Raphson. Despite using three subdomains, the resulting polynomial solution is of fourth degree, indicating improved accuracy. A comparison with other methods, such as MTSM from existing literature, underscores the effectiveness of employing MWR as a practical tool for achieving accurate solutions to boundary value problems, avoiding the need for more complex and cumbersome procedures.

DECLARATION OF INTERESTS STATEMENT

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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