

## Some Properties of Fractional Vector Analysis



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**ABSTRACT:** We are motivated to work on this topic by some research papers dealing with fractional derivatives. Using the notion of Conformable Fractional Derivative provided in R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh [5], we prove certain relevant features of Fractional Gradient, Divergence, and Curl in this study.

**KEYWORDS:** Conformable Fractional Derivative; Fractional Calculus; Fractional Differential Equations; Fractional gradient, divergence, curl.

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### 1. INTRODUCTION

Non-integer order derivatives and integrals are studied and applied in fractional calculus, a new discipline of mathematics. It has more than a 320-years history. Its progress has primarily been centred on the pure mathematical discipline. Liouville, Riemann, Leibniz, and others appear to have conducted the first systematic research in the 19th century. Fractional differential equations (FDEs) have been utilized to explain a variety of stable physical phenomena with anomalous degradation throughout the last two decades. Many mathematical models of real-world problems that arise in engineering and research are either linear or non-linear systems. With the discovery of fractional calculus, it has been shown that differential systems may be used to describe the role of many systems. It's worth noting that the fractional calculus may be used to describe a variety of physical processes with memory and genetic features. In reality, fractional order systems make up the majority of realworld processes.

Fractional Calculus is as old as Calculus. L' Hospital wrote to Leibniz on September 30th, 1695, inquiring about a particular notation he had used in his writings for the  $m$ th-derivative of the linear function  $f(x) = x, \frac{d^m x}{dx^m}$ .

L'Hopital's raised the question to Leibniz, what will happen if  $m = \frac{1}{2}$  ? Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn". As a result, fractional calculus was born on that date, and it is now known as fractional calculus' birthday. Many academics have attempted to define fractional derivative since then. The Riemann-Liouville and Caputo definitions are the most important. We recommend the reader to [1-4] for the history and important results on fractional derivatives.

**Definition 1.1 "Riemann-Liouville Definition":** For  $\beta \in [m - 1, m)$ ,  $\beta$ -derivative of function  $g(u)$  is

$$D_a^\beta(g)(u) = \frac{1}{\Gamma(m - \beta)} \frac{d^m}{dt^m} \int_a^u \frac{g(x)}{(u - x)^{\beta - m + 1}} dx.$$

**Definition 1.2 "Caputo Definition":** For  $\beta \in [m - 1, m)$ ,  $\beta$ -derivative of function  $g(u)$  is

$$D_a^\beta(g)(u) = \frac{1}{\Gamma(m - \beta)} \int_a^u \frac{g^m(x)}{(u - x)^{\beta - m + 1}} dx.$$

All of the definitions thus far, including 1.1 and 1.2, satisfy the property that the fractional derivative is linear. By all definitions, this is the single property inherited from the first derivative. However, such definitions have some drawbacks:

- The R-L derivative does not satisfy  $D_a(1) = 0$  ( $D_a^\beta(1) = 0$  for "Caputo definition") if  $\beta \notin \mathbb{N}$ .
- All fractional derivatives do not satisfy the known formulas of the derivative of the product and quotient of two functions:

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$$D_a^\beta(hg) = hD_a^\beta(g) + gD_a^\beta(h).$$

$$D_a^\beta\left(\frac{h}{g}\right) = \frac{gD_a^\beta(h) - hD_a^\beta(g)}{g^2} \quad \beta$$

iii. All fractional derivatives do not satisfy the Chain Rule:

$$D_a^\beta(hog)(t) = h^{(\beta)}(g(t))g^{(\beta)}(t).$$

iv. All fractional derivatives do not satisfy  $D_a^\alpha D_a^\beta(g) = D_a^{\alpha+\beta}(g)$  in general.

## 2. NOTATIONS AND PRELIMINARIES

The conformable fractional derivative, defined by the authors in [5] is a well-behaved simple fractional derivative that is based solely on the derivative's basic limit definition.

Khalil et. al. [5] have introduced a new derivative called "the conformable fractional derivative" of function

'g' of order ' $\beta$ ' and is defined by

$$T_\beta(g(u)) = \lim_{\epsilon \rightarrow 0} \frac{g(u+\epsilon u^{1-\beta}) - g(u)}{\epsilon},$$

where  $g: [0, \infty) \rightarrow \mathbb{R}$  and  $0 < \beta \leq 1$ , and the fractional derivative at 0 is defined as

$$g^{(\beta)}(0) = \lim_{t \rightarrow 0^+} T_\beta(g(u)).$$

This new definition satisfies:

1.  $T_\beta(af + bg) = aT_\beta(f) + bT_\beta(g)$  for all real a, b.

2.  $T_\beta(c) = 0$  for all constant functions  $f(t) = c$ .

Further, for  $0 < \beta \leq 1$ , and f, g be  $\beta$ -differentiable at apoint t, with  $(t) \neq 0$ , we have

3.  $T_\beta(fg) = fT_\beta(g) + gT_\beta(f)$ .

$$T_\beta\left(\frac{f}{g}\right) = \frac{gT_\beta(f) - fT_\beta(g)}{g^2}.$$

- 4.

Fractional derivatives of certain functions are given by:

1.  $T_\beta(t^p) = pt^{p-\beta}$ .

$$T_\beta\left(\sin \frac{1}{\beta} t^\beta\right) = \cos \frac{1}{\beta} t^\beta.$$

$$T_\beta\left(\cos \frac{1}{\beta} t^\beta\right) = -\sin \frac{1}{\beta} t^\beta$$

$$T_\beta\left(e^{\frac{1}{\beta} t^\beta}\right) = e^{\frac{1}{\beta} t^\beta}.$$

- 2.

- 3..

- 4.

On letting  $\beta = 1$  in these derivatives, we get the corresponding ordinary derivatives.

## 3. MAIN DEFINITIONS

The fractional gradient, divergence, and curl described by M. Mhailan, M. Abu Hammad, M. Al Horani, and R. Khalil [6] are defined in this section.

In this paper we use the notation  $D_x^\beta u$  to denote the conformable  $\beta$ - derivative of u with respect to the variable x, where u is function of several variables with domain  $\{(x, y, z): x, y, z > 0\}$ . SV will be used to represent such a space of functions. VF stands for the space of vector field  $F: R^3 \rightarrow R^3$ , with domain  $\{(x, y, z): x, y, z > 0\}$ .

**Definition 3.1[6]:** For  $f \in SV$ , Fractional Gradient is defined as  $\nabla^\beta f = D_x^\beta f i + D_y^\beta f j + D_z^\beta f k$

where  $\nabla^\beta = iD_x^\beta + jD_y^\beta + kD_z^\beta$ .

**Definition 3.2[6]:** For  $F = Pi + Qj + Rk \in VF$ , Fractional Divergence is defined as

$$\nabla^\beta \cdot F = D_x^\beta P + D_y^\beta Q + D_z^\beta R.$$

**Definition 3.3[6]:** For  $F = Pi + Qj + Rk \in VF$ , Fractional Curl is defined as

$$\nabla^\beta \times F = (D_y^\beta R - D_z^\beta Q)i + (D_x^\beta R - D_z^\beta P)j + (D_x^\beta Q - D_y^\beta P)k.$$

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### 4. MAIN RESULTS

In this section, we prove some useful properties of Fractional Gradient, Divergence and Curl using the definition of Conformable Fractional Derivative.

#### Theorem 4.1(Properties of Fractional Gradient):

- i.  $\nabla^\beta$  is linear. ii.  $\nabla^\beta f = 0$  if  $f$  is constant. iii.  
 $\nabla(fg) = f \nabla^\beta g + g \nabla^\beta f$ . iv.  
 $\nabla^\beta \left(\frac{f}{g}\right) = \frac{g \nabla^\beta f - f \nabla^\beta g}{g^2}$ .  
 v. If  $\vec{r}(\beta) = \frac{x^\beta}{\beta} i + \frac{y^\beta}{\beta} j + \frac{z^\beta}{\beta} k$ , then  $\nabla^\beta r(\beta) = \beta \frac{\vec{r}(\beta)}{r(\beta)}$ .  
 vi.  $\nabla^\beta r(\beta) = \beta^2 n r^{n-2} r^\rightarrow(\beta)$ .

**Proof:** The definition itself directly leads to parts (i) and (ii). Since they are essential, we pick to demonstrate (iii)-(vi).

$$\begin{aligned} \text{(iii)} \nabla^\beta(fg) &= (iD_x^\beta + jD_y^\beta + kD_z^\beta)fg \\ &= (iD_x^\beta fg + jD_y^\beta fg + kD_z^\beta fg) \\ &= i(fD_x^\beta g + gD_x^\beta f) + j(fD_y^\beta g + gD_y^\beta f) + k(fD_z^\beta g + gD_z^\beta f) \\ &= f(iD_x^\beta g + jD_y^\beta g + kD_z^\beta g) + g(iD_x^\beta f + jD_y^\beta f + kD_z^\beta f) \\ &= f \nabla^\beta g + g \nabla^\beta f. \end{aligned}$$

$$\begin{aligned} \text{(iv)} \nabla^\beta \left(\frac{f}{g}\right) &= (iD_x^\beta + jD_y^\beta + kD_z^\beta) \left(\frac{f}{g}\right) \\ &= \left(iD_x^\beta \frac{f}{g} + jD_y^\beta \frac{f}{g} + kD_z^\beta \frac{f}{g}\right) \\ &= i \left(\frac{g D_x^\beta f - f D_x^\beta g}{g^2}\right) + j \left(\frac{g D_y^\beta f - f D_y^\beta g}{g^2}\right) + k \left(\frac{g D_z^\beta f - f D_z^\beta g}{g^2}\right) \\ &= \frac{g(D_x^\beta f i + D_y^\beta f j + D_z^\beta f k) - f(D_x^\beta g i + D_y^\beta g j + D_z^\beta g k)}{g^2} \\ &= \frac{g}{g^2} \nabla^\beta f - \frac{f}{g^2} \nabla^\beta g \end{aligned}$$

$$\begin{aligned} \text{(v)} \text{ If } \vec{r}(\beta) = \frac{x^\beta}{\beta} i + \frac{y^\beta}{\beta} j + \frac{z^\beta}{\beta} k, \text{ then } \nabla^\beta r(\beta) &= (iD_x^\beta + jD_y^\beta + kD_z^\beta) \frac{r(\beta)}{\beta} \\ &= \frac{1}{\beta} (iD_x^\beta + jD_y^\beta + kD_z^\beta) r(\beta) \\ &= \frac{1}{\beta} \left( i \frac{\beta}{r(\beta)} x^\beta + j \frac{\beta}{r(\beta)} y^\beta + k \frac{\beta}{r(\beta)} z^\beta \right) \\ &= \frac{1}{\beta} \left( i \frac{\beta}{r(\beta)} x^\beta + j \frac{\beta}{r(\beta)} y^\beta + k \frac{\beta}{r(\beta)} z^\beta \right) \\ &= \beta \frac{\vec{r}(\beta)}{r(\beta)} \end{aligned}$$

provided partial derivatives are performed using the definition of Conformable Fractional Derivative.

If  $\beta = 1$ , then we get  $grad(r) = \frac{\vec{r}}{r}$ .

$$\begin{aligned} \text{(vi)} \nabla^\beta r^n(\beta) &= iD_x^\beta r^n(\beta) + jD_y^\beta r^n(\beta) + kD_z^\beta r^n(\beta) \\ &= i \left[ x^{1-\beta} n r^{n-1} \frac{\partial r}{\partial x} \right] + j \left[ y^{1-\beta} n r^{n-1} \frac{\partial r}{\partial y} \right] + k \left[ z^{1-\beta} n r^{n-1} \frac{\partial r}{\partial z} \right] \\ &= [\beta x^\beta n r^{n-2}] + j[\beta y^\beta n r^{n-2}] + k[\beta z^\beta n r^{n-2}] \\ &= \beta^2 n r^{n-2} r^\rightarrow(\beta). \end{aligned}$$

If  $\beta = 1$ , then we get  $grad r^n = n r^{n-2} r^\rightarrow$ .

#### Theorem 4.2(Properties of Fractional Divergence):

- i. Fractional Divergence is linear. ii.  $\nabla^\beta \cdot \mathbf{F} = 0$  if  $\mathbf{F}$  is constant. iii.  $\nabla^\beta \cdot (f\mathbf{F}) = \mathbf{F} \cdot \nabla^\beta f + f \nabla^\beta \cdot \mathbf{F}$  where  $f \in SV$  and  $\mathbf{F} \in VF$ .

**Proof:** The definition itself directly leads to parts (i) and (ii). Since this is essential, we decide to demonstrate (iii).

$$\begin{aligned} \text{(iii)} \text{ For } F = Pi + Qj + Rk \in VF, \nabla^\beta \cdot (f\mathbf{F}) &= (iD_x^\beta + jD_y^\beta + kD_z^\beta) \cdot (f\mathbf{F}) \\ &= (iD_x^\beta + jD_y^\beta + kD_z^\beta) \cdot (f\mathbf{F}) \\ &= \left( i \cdot D_x^\beta (f\mathbf{F}) + j \cdot D_y^\beta (f\mathbf{F}) + k \cdot D_z^\beta (f\mathbf{F}) \right) \\ &= \sum i \cdot D_x^\beta (f\mathbf{F}) \end{aligned}$$

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$$\begin{aligned}
 &= \sum i . (f D_x^\beta \mathbf{F} + D_x^\beta f \mathbf{F}) \\
 &= f \sum (i . D_x^\beta \mathbf{F}) + \sum (i D_x^\beta f) . \mathbf{F} \\
 &= \mathbf{F} . \nabla^\beta f + f \nabla^\beta . \mathbf{F}
 \end{aligned}$$

### Theorem 4.3(Properties of Fractional Curl):

- i. Fractional Curl is linear.
- ii.  $\nabla^\beta \times \mathbf{F} = 0$  if  $\mathbf{F}$  is constant. iii.  $\nabla^\beta \times (f\mathbf{F}) = \nabla^\beta f \times \mathbf{F} + f \nabla^\beta \times \mathbf{F}$  where  $f \in SV$  and  $\mathbf{F} \in VF$ . iv.  $\nabla^\beta . (\mathbf{F} \times \mathbf{G}) = \mathbf{G} . (\nabla^\beta \times \mathbf{F}) - \mathbf{F} . (\nabla^\beta \times \mathbf{G})$ .

Proof: The definition itself directly leads to parts (i) and (ii). Since they are essential, we pick to demonstrate (iii) and (iv).

$$\begin{aligned}
 \text{(iii) } \nabla^\beta \times (f\mathbf{F}) &= (i D_x^\beta + j D_y^\beta + k D_z^\beta) \times (f\mathbf{F}) \\
 &= \sum i \times D_x^\beta (f\mathbf{F}) \\
 &= \sum i \times (D_x^\beta f \mathbf{F} + f D_x^\beta \mathbf{F}) \\
 &= \sum (i \times D_x^\beta f \mathbf{F}) + f \sum (i \times D_x^\beta \mathbf{F}) \\
 &= \sum D_x^\beta f (i \times \mathbf{F}) + f \sum (i \times D_x^\beta \mathbf{F}) \qquad = \nabla^\beta f \times \mathbf{F}
 \end{aligned}$$

+  $f \nabla^\beta \times \mathbf{F}$

$$\begin{aligned}
 \text{(iv) } \nabla^\beta . (\mathbf{F} \times \mathbf{G}) &= \sum i . D_x^\beta (\mathbf{F} \times \mathbf{G}) \\
 &= \sum i . (D_x^\beta \mathbf{F} \times \mathbf{G} + \mathbf{F} \times D_x^\beta \mathbf{G}) \\
 &= \sum i . (D_x^\beta \mathbf{F} \times \mathbf{G}) + \sum i . (\mathbf{F} \times D_x^\beta \mathbf{G}) \\
 &= \sum i . (D_x^\beta \mathbf{F} \times \mathbf{G}) - \sum i . (D_x^\beta \mathbf{G} \times \mathbf{F}) \\
 &= \sum (i \times D_x^\beta \mathbf{F}) . \mathbf{G} - \sum (i \times D_x^\beta \mathbf{G}) . \mathbf{F} \\
 &= (\nabla^\beta \times \mathbf{F}) . \mathbf{G} - (\nabla^\beta \times \mathbf{G}) . \mathbf{F} \qquad = \mathbf{G} . (\nabla^\beta \times \mathbf{F}) - \mathbf{F} . (\nabla^\beta \times \mathbf{G})
 \end{aligned}$$

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